

Curvature and Gravity Actions for Matrix Models

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Abstract

We show how gravitational actions, in particular the Einstein-Hilbert action, can be obtained from additional terms in Yang-Mills matrix models. This is consistent with recent results on induced gravitational actions in these matrix models, realizing space-time as 4-dimensional brane solutions. It opens up the possibility for a controlled non-perturbative description of gravity through simple matrix models, with interesting perspectives for the problem of vacuum energy. The relation with UV/IR mixing and non-commutative gauge theory is discussed.

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1 Background

For a long time people have tried to combine the ideas of Quantum Mechanics and General Relativity in a consistent manner. In fact, such a combination strongly suggests a quantum structure of space-time itself near the Planck scale. While some aspects of such a quantum space-time can be seen in string theory or loop quantum gravity, a satisfactory understanding is still missing. A different approach to this problem has been discussed in recent years: Classical space-time is replaced by a quantized or non-commutative (NC) space, where the coordinate functions x^μ are replaced by matrices resp. Hermitian operators X^μ acting on a Hilbert space \mathcal{H} , which satisfy some non-trivial commutation relations

$$[X^\mu, X^\nu] = i\theta^{\mu\nu} . \quad (1)$$

While the simplest case of a Heisenberg algebra, i.e. with constant commutator $\theta^{\mu\nu}$, leads to non-commutative field theories (cf. [1–4] for a review of the topic), a dynamical commutator seems essential in the context of gravity. At the semi-classical level, the commutation relations (1) determine a Poisson structure $\theta^{\mu\nu}$ on space-time, which is expected to be dynamical.

The approach we would like to take is to consider matrix models of Yang-Mills type, which have already been shown to incorporate gravity, at least at the semi-classical level [5–7]. In this spirit, we follow Ref. [8] by considering the matrix model action

$$S_{YM} = -\text{Tr}[X^a, X^b][X^c, X^d]\eta_{ac}\eta_{bd} , \quad (2)$$

where η_{ac} is the (flat) metric of a D dimensional embedding space (i.e. $a, b, c, d \in 1, \dots, D$). It can be purely Euclidean, or have one or more time-like directions. The “covariant coordinates” X^a are Hermitian matrices, resp. operators acting on a separable Hilbert space \mathcal{H} . We denote the commutator of two coordinates as

$$[X^a, X^b] = i\theta^{ab} . \quad (3)$$

Furthermore, we consider for simplicity¹ configurations where some of the X^a are functions of the remaining ones, because we are interested in $2n$ dimensional non-commutative spaces \mathcal{M}_θ^{2n} . We can then split the matrices resp. coordinates as

$$X^a = (X^\mu, \phi^i), \quad \mu = 1, \dots, 2n, \quad i = 1, \dots, D - 2n, \quad (4)$$

so that the $\phi^i(X) \sim \phi^i(x)$ define in the semi-classical limit an embedding of a $2n$ dimensional submanifold

$$\mathcal{M}^{2n} \hookrightarrow \mathbb{R}^D. \quad (5)$$

Moreover, we can interpret

$$[X^\mu, X^\nu] \sim i\theta^{\mu\nu}(x) \quad (6)$$

in the semi-classical limit as a Poisson structure on \mathcal{M}^{2n} . Thus we are considering quantized Poisson manifolds $(\mathcal{M}^{2n}, \theta^{\mu\nu})$, with quantized embedding functions X^a . Throughout this paper, \sim denotes the semi-classical limit, where commutators are replaced by Poisson brackets. We will assume that $\theta^{\mu\nu}$ is non-degenerate, so that its inverse matrix $\theta_{\mu\nu}^{-1}$ defines a symplectic form on \mathcal{M}^{2n} . The sub-manifold $\mathcal{M}^{2n} \subset \mathbb{R}^D$ is equipped with a non-trivial induced metric

$$g_{\mu\nu}(x) = \partial_\mu x^a \partial_\nu x^b \eta_{ab} = \eta_{\mu\nu} + \partial_\mu \phi^i \partial_\nu \phi^j \eta_{ij}, \quad (7)$$

via pull-back of η_{ab} . Finally, we define the following quantities [8]:

$$\begin{aligned} G^{\mu\nu} &= e^{-\sigma} \theta^{\mu\rho} \theta^{\nu\sigma} g_{\rho\sigma}, & \eta &= \frac{1}{4} e^\sigma G^{\mu\nu} g_{\mu\nu}, \\ \rho &= \sqrt{\det \theta_{\mu\nu}^{-1}}, & e^{-\sigma} &= \frac{\rho}{\sqrt{\det G_{\mu\nu}}}. \end{aligned} \quad (8)$$

The last relation gives a unique definition for $e^{-\sigma}$ provided $n > 1$, which we assume. Of particular interest is the special case where $2n = 4$ and

$$G^{\mu\nu} = g^{\mu\nu} \quad \rightarrow \quad \eta = e^\sigma, \quad (9)$$

which corresponds to $\theta^{\mu\nu}$ being self-dual with respect to the metric $g_{\mu\nu}$ (cf. [10]).

In order to understand the effective geometry of \mathcal{M}^{2n} , consider a test-particle on \mathcal{M}^{2n} , modeled by a scalar field φ for simplicity (this could be e.g. an $\mathfrak{su}(k)$ component of ϕ^i). In order to preserve gauge invariance, the kinetic term must have the form

$$\begin{aligned} S[\varphi] &\equiv -\text{Tr}[X^a, \varphi][X^b, \varphi] \eta_{ab} \sim \frac{1}{(2\pi)^n} \int d^{2n}x \frac{1}{|\theta^{\mu\nu}|^{1/2}} e^a(\varphi) e^b(\varphi) \eta_{ab} \\ &= \frac{1}{(2\pi)^n} \int d^{2n}x \frac{1}{|\theta^{\mu\nu}|^{1/2}} \theta^{\mu\mu'}(x) \theta^{\nu\nu'}(x) g_{\mu\nu} \partial_{\mu'} \varphi \partial_{\nu'} \varphi \\ &= \frac{1}{(2\pi)^n} \int d^{2n}x |G_{\mu\nu}|^{1/2} G^{\mu\nu}(x) \partial_\mu \varphi \partial_\nu \varphi, \end{aligned} \quad (10)$$

denoting the D natural vector fields on \mathcal{M}^{2n} defined by the matrix model as

$$e^a(f) := -i[X^a, f] \sim \theta^{\mu\nu} \partial_\mu x^a \partial_\nu f, \quad (11)$$

¹However, this framework is not restricted to spaces with trivial topology, as can be seen e.g. by the example of a fuzzy sphere [9].

in the semi-classical limit where commutators are replaced by Poisson brackets. Therefore the kinetic term for φ on \mathcal{M}^{2n} is governed by the effective metric $G_{\mu\nu}(x)$, which depends on the Poisson tensor $\theta^{\mu\nu}(x)$ and the embedding metric $g_{\mu\nu}(x)$. In fact, the same metric also governs non-Abelian gauge fields and fermions in the matrix model (up to possible conformal factors), so that $G_{\mu\nu}$ *must* be interpreted as gravitational metric. There is no need and no room for invoking any “principles”. Since the embedding ϕ^i is dynamical, the model describes a dynamical theory of gravity, realized on dynamically determined submanifolds of \mathbb{R}^D .

We furthermore note that

$$|G_{\mu\nu}(x)| = |g_{\mu\nu}(x)|, \quad 2n=4 \quad (12)$$

which means that in the 4-dimensional case, the Poisson tensor $\theta^{\mu\nu}$ does not enter the Riemannian volume at all. This turns out to stabilize flat space, and is one of several reasons why 4 dimensions are special in this framework.

Equations of motion The bare matrix model Eqn. (2) without matter leads to the following e.o.m. for X^c :

$$[X^a, [X^b, X^c]]\eta_{ab} = 0. \quad (13)$$

It was shown in Ref. [8] that in the semi-classical limit, these equations can be brought into the covariant form using Eqn. (71b)

$$\square_G \phi^i = 0, \quad (14a)$$

$$\square_G x^\mu = 0, \quad (14b)$$

which imply

$$\nabla_G^\mu (e^\sigma \theta_{\mu\nu}^{-1}) = G_{\mu\nu} \theta^{\mu\rho} e^{-\sigma} \partial_\rho (e^\sigma \eta). \quad (15)$$

Here ∇_G denotes the Levi-Civita connection with respect to the effective metric $G^{\mu\nu}$, and \square_G denotes the corresponding Laplace-Beltrami operator. Eqn. (15) provides the relation between the non-commutativity $\theta^{\mu\nu}(x)$ and the metric $G^{\mu\nu}$. Since it essentially has the form of covariant Maxwell equations coupled to an external current, it will have a unique solution for suitable “boundary conditions”

$$\theta_{\mu\nu}(x) \rightarrow \bar{\theta}_{\mu\nu} = \text{const} \quad \text{for } |x| \rightarrow \infty, \quad (16)$$

up to radiative contributions.

Ward identity For the matrix model (2) one can derive the “energy-momentum tensor”

$$T^{ab} = \frac{1}{2} [[X^a, X^c], [X^b, X^{c'}]] + \eta_{cc'} - \frac{1}{4} \eta^{ab} [X^c, X^{d'}] [X^{c'}, X^{d'}] \eta_{cc'} \eta_{dd'}, \quad (17)$$

whose conservation follows directly from the matrix equations of motion (13) above:

$$[X^a, T^{a'b}] \eta_{aa'} = 0. \quad (18)$$

In fact, this conservation constitutes a matrix Ward identity (cf. [8, 10]) corresponding to the infinitesimal transformation

$$\delta X^a = [X^b, [X^a, \epsilon_b]]_+, \quad (19)$$

for arbitrary matrices $\epsilon_b = \epsilon_b(X)$. Eqn. (19) defines a measure-preserving infinitesimal transformation on the space of matrices which satisfies

$$\delta S = -8\text{Tr}\epsilon_b[X^a, T^{a'b}]\eta_{aa'}. \quad (20)$$

This implies (18) since the ϵ_b are arbitrary.

2 Geometric considerations

Since space-time is described in the matrix model as 4-dimensional submanifold $\mathcal{M}^4 \subset \mathbb{R}^D$, one should expect that both the intrinsic as well as the extrinsic geometry of \mathcal{M}^4 will play some role. In this section, we provide the necessary tools for an efficient description of the geometry and the intrinsic (Riemannian) curvature of such branes; for a related discussion see e.g. [11]. We restrict ourselves mostly to 4-dimensional configurations with

$$G_{\mu\nu} = g_{\mu\nu}. \quad (21)$$

One can easily see² that for 4-dimensional spaces \mathcal{M}^4 , this is equivalent to the symplectic form

$$\omega = \frac{1}{2}\theta_{\mu\nu}^{-1}dx^\mu \wedge dx^\nu \quad (22)$$

being (anti-) self-dual³, $\star\omega = \pm\omega$. This imposes no significant restriction on the effective geometry $G_{\mu\nu} = g_{\mu\nu}$, since such (anti)self-dual ω can essentially always be found for a given metric (assuming e.g. that \mathcal{M}^4 is globally hyperbolic). In that case $\eta = e^\sigma$ (cf. (8)), and Eqn. (15) reduces to

$$\nabla^\mu \theta_{\mu\nu}^{-1} = 0, \quad (23)$$

which is satisfied identically for self-dual ω . We can drop the subscripts g or G to the covariant derivatives from now on.

Under this assumption, the conserved “tensor” T^{ab} acquires a simple geometrical meaning in the semi-classical limit: it essentially becomes the projector on the normal space $N_p\mathcal{M}^4 = (T_p\mathcal{M}^4)^\perp$. More precisely,

$$\begin{aligned} T^{ab} &\sim e^\sigma \mathcal{P}_N^{ab}, \\ \mathcal{P}_T^{ab} &= g^{\mu\nu} \partial_\mu x^a \partial_\nu x^b, \quad \mathcal{P}_N^{ab} = \eta^{ab} - \mathcal{P}_T^{ab}, \end{aligned} \quad (24)$$

where $\mathcal{P}_{N,T}$ are the projectors on the normal resp. tangential space at $p \in \mathcal{M}^4$. This means that

$$\begin{aligned} \mathcal{P}_T^{ab} \eta_{bc} \partial_\mu x^c &= \partial_\mu x^a, & \mathcal{P}_N^{ab} \eta_{bc} \partial_\mu x^c &= 0, \\ \mathcal{P}_T^2 &= \mathcal{P}_T, & \mathcal{P}_N^2 &= \mathcal{P}_N, \end{aligned} \quad (25)$$

²by going to local coordinates where $g_{\mu\nu}$ is diagonal at a point and $\theta^{\mu\nu}$ has canonical form [10]

³In the case of Minkowski signature, time-like matrices X^0 should be anti-hermitian as explained in [10]. We then adopt the convention that ε^{0123} is imaginary, so that $\star^2 = 1$.

which is easy to verify. Here

$$\partial_\mu \equiv (\partial_\mu x^a)_{a=1,2,\dots,D} \in T_p \mathcal{M}^4 \subset \mathbb{R}^D \quad (26)$$

is interpreted as tangent vector field defined by some coordinate system, represented in D -component notation. This allows to write down covariant derivatives $\nabla \equiv \nabla_g$ with respect to the embedding metric. For example, the covariant derivative of the vector field ∂_μ defined by some coordinate system (i.e. $V = V_{(\mu)}^\nu \partial_\nu$ with $V_{(\mu)}^\nu = \delta_\mu^\nu$) is

$$(\nabla_\mu \partial_\nu)^a = \mathcal{P}_T^{ab} \eta_{bc} \partial_\mu \partial_\nu x^c = \left(g^{\lambda\sigma} \partial_\sigma x^b \eta_{bc} \partial_\mu \partial_\nu x^c \right) \partial_\lambda x^a = \Gamma_{\mu\nu}^\lambda (\partial_\lambda)^a, \quad (27)$$

where

$$\Gamma_{\mu\nu}^\lambda = g^{\lambda\sigma} \partial_\sigma x^b \eta_{bc} \partial_\mu \partial_\nu x^c \quad (28)$$

is the Christoffel symbol w.r.t. $g_{\mu\nu}$. From this it is easy to recover the standard formula in terms of $g_{\mu\nu}$. Notice that using the D -dimensional Poincare symmetry, we can choose for any given point $p \in \mathcal{M}^4$ the matrix coordinates $x^a = (x^\mu, \phi^i)$ such that $\partial_\mu \phi^i|_p = 0$. These are called matrix normal coordinates, which satisfy $\Gamma_{\mu\nu}^\lambda|_p = 0$. Similarly, the second covariant derivative of the scalar fields defined by the matrices $x^a : \mathcal{M}^4 \rightarrow \mathbb{R}^D$ (i.e. the second fundamental form) is given by⁴

$$\nabla_\mu \nabla_\nu x^a = \mathcal{P}_N^{ab} \eta_{bc} \partial_\mu \partial_\nu x^c = (\partial_\mu \partial_\nu - \Gamma_{\mu\nu}^\rho \partial_\rho) x^a. \quad (29)$$

It immediately follows that

$$\nabla_\mu x^a \nabla_\nu \nabla_\rho x_a = 0, \quad (30)$$

which can also be seen from $\nabla g = 0$ or by going to normal embedding coordinates. Here and in the following we adopt the convention that Latin indices of $X^a \sim x^a$ are raised or lowered with the constant D -dimensional background metric η_{ab} . It follows that

$$\mathcal{P}_N^{ab} \nabla_\mu \nabla_\nu x_b = \nabla_\mu \nabla_\nu x^a \quad (31)$$

which will be used below.

We can now write down the Riemann curvature tensor in a useful form. Consider

$$\begin{aligned} (-R_{\nu\mu\lambda}{}^\kappa \partial_\kappa)^a &= ([\nabla_\nu, \nabla_\mu] \partial_\lambda)^a \\ &= (\mathcal{P}_T^{ab} \partial_\nu \mathcal{P}_T^{b'c} \partial_\mu - \mathcal{P}_T^{ab} \partial_\mu \mathcal{P}_T^{b'c} \partial_\nu) \partial_\lambda x_c \eta_{bb'} \\ &= \mathcal{P}_T^{ab} (\partial_\nu \mathcal{P}_T^{b'c} \partial_\mu - \partial_\mu \mathcal{P}_T^{b'c} \partial_\nu) \partial_\lambda x_c \eta_{bb'} \\ &= \partial_\kappa x^a \left(\partial^\kappa x^b \partial_\nu \mathcal{P}_T^{b'c} \partial_\mu \partial_\lambda x_c - \partial^\kappa x^b \partial_\mu \mathcal{P}_T^{b'c} \partial_\nu \partial_\lambda x_c \right) \eta_{bb'}, \end{aligned} \quad (32)$$

hence

$$\begin{aligned} R_{\nu\mu\lambda\kappa} &= -\partial_\nu \mathcal{P}_T^{bc} \partial_\kappa x_b \partial_\mu \partial_\lambda x_c + \partial_\mu \mathcal{P}_T^{bc} \partial_\kappa x_b \partial_\nu \partial_\lambda x_c, \\ R_{\nu\lambda} &= R_{\nu\mu\lambda}{}^\mu = -\partial_\nu \mathcal{P}_T^{bc} \partial^\mu x_b \partial_\mu \partial_\lambda x_c + \partial_\mu \mathcal{P}_T^{bc} \partial^\mu x_b \partial_\nu \partial_\lambda x_c, \\ R &= R_{\nu\lambda} g^{\nu\lambda} = \partial_\nu \mathcal{P}_T^{bc} \left(-\partial^\mu x_b \partial_\mu \partial_\lambda x_c g^{\lambda\nu} + \partial^\nu x_b \partial_\mu \partial_\lambda x_c g^{\lambda\mu} \right). \end{aligned} \quad (33)$$

⁴Notice the difference to (27), where ∂_μ is interpreted as a vector field.

Using (25) and noting $\partial_\nu \mathcal{P}_T^{bc} \partial_\kappa x_b = \partial_\nu (\mathcal{P}_T^{bc} \partial_\kappa x_b) - \mathcal{P}_T^{bc} \partial_\nu \partial_\kappa x_b$ this can be written as

$$\begin{aligned} R_{\nu\mu\lambda\kappa} &= -\mathcal{P}_N^{ab} (\partial_\kappa \partial_\nu x_a \partial_\lambda \partial_\mu x_b - \partial_\kappa \partial_\mu x_a \partial_\nu \partial_\lambda x_b) \\ &= -\nabla_\kappa \nabla_\nu x^a \nabla_\lambda \nabla_\mu x_a + \nabla_\kappa \nabla_\mu x^a \nabla_\nu \nabla_\lambda x_a \end{aligned} \quad (34)$$

which is nothing else but the Gauss-Codazzi theorem. In particular, the Ricci scalar is given by

$$\begin{aligned} R &= -\mathcal{P}_N^{ab} (\partial_\kappa \partial_\nu x_a \partial_\lambda \partial_\mu x_b - \partial_\kappa \partial_\mu x_a \partial_\nu \partial_\lambda x_b) g^{\kappa\mu} g^{\lambda\nu} \\ &= -\nabla_\mu \nabla_\nu x^a \nabla^\mu \nabla^\nu x_a + \square_g x^a \square_g x_a. \end{aligned} \quad (35)$$

Finally, we note that the conservation law (18) in the semi-classical limit reduces to

$$0 = \theta^{\mu\nu} \partial_\mu x_b \partial_\nu T^{bc} = \theta^{\mu\nu} \partial_\nu (\partial_\mu x_b T^{bc}). \quad (36)$$

This holds as long as T^{bc} is a projector on the normal bundle (see Eqns. (24) and (25)), which follows from $g_{\mu\nu} = G_{\mu\nu}$.

3 Extensions to the matrix model action

In this section we would like to discuss some possible extensions to the matrix model action (2). In particular, we will find terms which depend only on the intrinsic geometry of $\mathcal{M}^4 \subset \mathbb{R}^D$, including essentially the Einstein-Hilbert action as well as a term coupling the Riemann tensor to the Poisson tensor. This allows to realize Einstein gravity (in a slightly modified form) and its quantization through matrix models. The terms obtained are in agreement with the one-loop effective action for the gravitational sector of the Yang-Mills matrix model [8, 12]. This shows that the quantization of the model can be addressed both from the geometrical point of view as well as from the matrix model point of view. It opens up the possibility for a controlled non-perturbative quantization of the matrix model by adding suitable counter terms.

Throughout this section we consider 4-dimensional $\mathcal{M}^4 \subset \mathbb{R}^D$ with self-dual $\theta_{\mu\nu}^{-1}$, i.e. $G_{\mu\nu} = g_{\mu\nu}$.

3.1 Order 6 terms

We first note the following identities:

$$\begin{aligned} \text{Tr} \square X^a \square X_a &= \text{Tr} \left(\frac{1}{2} [X^c, [X^a, X^b]] [X_c, [X_a, X_b]] - 2 [X^a, X^c] [X_c, X^b] [X_a, X_b] \right), \\ \text{Tr} [X^a, [X^b, X^c]] [X_c, [X_a, X_b]] &= -\frac{1}{2} \text{Tr} [X^c, [X^a, X^b]] [X_c, [X_a, X_b]], \end{aligned} \quad (37)$$

using the abbreviation

$$\square X^a \equiv [X^b, [X_b, X^a]]. \quad (38)$$

This leaves the following independent terms:

$$S_6 = \text{Tr} \left(\alpha \square X^a \square X_a + \frac{\beta}{2} [X^c, [X^a, X^b]] [X_c, [X_a, X_b]] \right). \quad (39)$$

It turns out that these terms have a nice geometrical meaning in the semi-classical limit. As shown in Appendix A, one finds

$$S_6 \sim \frac{\alpha + \beta}{(2\pi)^2} \int d^4x \sqrt{g} e^\sigma \square_g x^a \square_g x_a + \frac{\beta}{(2\pi)^2} \int d^4x \sqrt{g} \left(\frac{1}{2} \theta^{\mu\rho} \theta^{\eta\alpha} R_{\mu\rho\eta\alpha} - 2e^\sigma R + 2e^\sigma \partial^\mu \sigma \partial_\mu \sigma \right) \quad (40)$$

using (35). The first term depends on the “extrinsic” geometry, i.e. the embedding $\mathcal{M}^4 \subset \mathbb{R}^D$, and vanishes for harmonic embeddings where $\square_g x^a = 0$. However, for $\alpha + \beta = 0$ the result is purely tensorial and intrinsic, independent of the particular embedding of $\mathcal{M}^4 \subset \mathbb{R}^D$. This is an essential feature of General Relativity. We will see that such terms are also induced at one loop when coupling fermions to the matrix model [12], hence the above terms can be used to cancel unwanted terms in the quantum effective action. The Einstein-Hilbert action is obtained from similar higher-order terms, as we show next.

3.2 Higher order terms

There are other terms in the matrix model which involve up to 4 derivatives in the semi-classical limit. Rather than giving an exhaustive list we only discuss some terms of particular interest here.

Order X^{10} terms. Consider the following order 10 terms in the semi-classical limit:

$$\begin{aligned} S_{10} &= (2\pi)^2 \text{Tr} \left([X^a, T^{bc}] [X_a, T_{bc}] + 2T^{ab} \square X_b \square X_c \right) \\ &\sim \int d^4x \sqrt{g} [(D - 2n) e^\sigma \square_g e^\sigma + 2e^{2\sigma} R] \end{aligned} \quad (41a)$$

$$\begin{aligned} \tilde{S}_{10} &= (2\pi)^2 \text{Tr} [X^a, X^b] [X_a, X_b] \square \left([X^c, X^d] [X_c, X_d] \right) \\ &\sim -16 \int d^4x \sqrt{g} e^\sigma \square_g e^\sigma, \end{aligned} \quad (41b)$$

where once more D is the dimension of the embedding space, and $2n = 4$ denotes the dimension of the non-commutative subspace.

Proof The semi-classical limit of the first term of (41a) is given by

$$\begin{aligned} (2\pi)^2 \text{Tr} [X^a, T^{bc}] [X_a, T_{bc}] &\sim - \int d^4x \sqrt{g} \partial_\mu (e^\sigma P_N^{bc}) \partial^\mu (e^\sigma (\eta_{bc} - \partial^\nu x_b \partial_\nu x_c)) \\ &= - \int d^4x \sqrt{g} \nabla_\mu (e^\sigma P_N^{bc}) \nabla^\mu (e^\sigma \eta_{bc} - e^\sigma \partial^\nu x_b \partial_\nu x_c) \\ &= \int d^4x \sqrt{g} e^\sigma P_N^{bc} \left(\eta_{bc} \nabla_\mu \nabla^\mu e^\sigma - \nabla_\mu \nabla^\mu (e^\sigma \partial^\nu x_b \partial_\nu x_c) \right) \\ &= \int d^4x \sqrt{g} e^\sigma \left((D - 2n) \square_g e^\sigma - 2P_N^{bc} (e^\sigma \nabla^\mu \partial^\nu x_b \nabla_\mu \partial_\nu x_c) \right) \\ &= \int d^4x \sqrt{g} e^\sigma \left((D - 4) \square_g e^\sigma - 2e^\sigma \nabla^\mu \partial^\nu x^a \nabla_\mu \partial_\nu x_a \right) \end{aligned} \quad (42)$$

using the properties (25) and (29) of the projector \mathcal{P}_N^{bc} .
The second term of Eqn. (41a) semi-classically is

$$\begin{aligned} (2\pi)^2 \text{Tr } T^{ab} \square X_b \square X_c &\sim \int d^4x \sqrt{g} e^{2\sigma} \mathcal{P}_N^{bc} \square_g x_b \square_g x_c \\ &= \int d^4x \sqrt{g} e^{2\sigma} \square_g x^c \square_g x_c \end{aligned} \quad (43)$$

and using (35) one finally arrives at Eqn. (41a). The second identity (41b) simply follows from

$$e^\sigma \Big|_{G=g} = \eta = \frac{1}{4} \theta^{\mu\nu} \theta^{\rho\sigma} g_{\mu\rho} g_{\nu\sigma} = \frac{1}{4} \{x^a, x^b\} \{x_a, x_b\}, \quad (44)$$

where $\{x^a, x^b\}$ denotes the Poisson bracket. ■

This means that the matrix model action⁵

$$\begin{aligned} S_{\text{E-H}} &= \text{Tr} \left([X^a, T^{bc}] [X_a, T_{bc}] + 2T^{ab} \square X_b \square X_c + \frac{D-4}{16} [X^a, X^b] [X_a, X_b] \square ([X^c, X^d] [X_c, X_d]) \right) \\ &= \text{Tr} \left(2T^{ab} \square X_a \square X_b - T^{ab} \square H_{ab} \right) \\ &\sim \frac{2}{(2\pi)^2} \int d^4x \sqrt{g} e^{2\sigma} R, \end{aligned} \quad (45)$$

where (cp. (17))

$$H^{ab} = \frac{1}{2} [[X^a, X^c], [X^b, X_c]]_+, \quad (46)$$

reduces in the semi-classical limit to the Einstein-Hilbert action, with an additional factor $e^{2\sigma}$ which introduces the required scale. After introducing an explicit dimensionful parameter Λ_0 of dimension length^{-1} (so that $X^a \sim x^a$ acquires the appropriate dimension of length) to the matrix model, we can identify the gravitational constant arising from this term as

$$\Lambda_{\text{planck}}^2 = G = \frac{\Lambda_0^{10}}{\Lambda_{NC}^8} \quad (47)$$

recalling that $e^\sigma = \Lambda_{NC}^{-4}$ sets the non-commutativity scale. It is thus reasonable to set $\Lambda_{NC} \sim \Lambda_0 \sim \Lambda_{\text{planck}}$. However, this should be taken with some caution since quantum effects will play an important role, as indicated below.

It is remarkable that one obtains in this way an action which depends only on the intrinsic geometry of $\mathcal{M}^4 \subset \mathbb{R}^D$. In particular every Ricci-flat manifold can be obtained as a solution of the matrix model with the term (45), for selfdual ω . Notice that self-dual ω is then indeed a solution, since $\int d^4x \sqrt{g} \delta e^{2\sigma} R = 0$. As an example, we will present an explicit realization of the Schwarzschild-solution in [13].

⁵In fact, it is easy to see that the equality of the second and third line of (45) holds even without the trace, resp. the integral.

Order X^8 terms The simplest term of order X^8 is given by

$$S_8 = \frac{(2\pi)^2}{4} \text{Tr}([X^a, X^b][X_b, X_c][X^c, X^d][X_d, X_a]). \quad (48)$$

The semi-classical limit of this term is obtained easily using (11), (7) and (8) for the self-dual case, i.e. $G = g$ and hence $\eta = e^\sigma$:

$$S_8 \sim \int d^4x \sqrt{g} e^\sigma. \quad (49)$$

A preliminary analysis suggests that the only other non-vanishing terms of order X^8 lead to higher-order derivative terms, such as $\text{Tr}[X, [X, \theta]][X, [X, \theta]]$ where θ stands for $[X, X]$. Such higher-derivative terms are typically suppressed at low energies and should be studied elsewhere.

3.3 Potentials

We now consider the possibility to add explicit "potential" terms to the matrix model which break the translational invariance $X^a \rightarrow X^a + c^a \mathbb{1}$, but preserve the D -dimensional rotational invariance of the matrix model. The simplest such extension to the matrix model is a "mass" like term of the form

$$S_m = (2\pi)^2 m^2 \text{Tr} X^a X_a \sim m^2 \int d^4x \sqrt{g} e^{-\sigma} x^a x_a. \quad (50)$$

Similarly, one could also add higher powers of $X^a X_a \sim x^2$ to the action, for example

$$S_{V_4} = (2\pi)^2 \mu_4 \text{Tr} \left(X^a X^b X^c X^d (\eta_{ab} \eta_{cd} + \eta_{ac} \eta_{bd}) \right) \sim 2\mu_4 \int d^4x \sqrt{g} e^{-\sigma} (x^2)^2. \quad (51)$$

Now consider the equation of motion in the presence of such potential terms. For example, upon adding a "mass" term $\text{Tr} V(X)$ with $V(X) = m^2 X^a X_a$, the e.o.m. of the matrix model becomes

$$\square X^a = \frac{1}{2} m^2 X^a. \quad (52)$$

Using (30), this implies semi-classically that

$$0 = \partial_\mu x^a \square_G x^a = \frac{1}{2} m^2 e^{-\sigma} \partial_\mu x^a x^a = \frac{1}{4} e^{-\sigma} \partial_\mu V(x) \quad (53)$$

which holds also for more general potentials $V(X) \sim V(x)$. This means that $V(x) = \text{const.}$, so that $\mathcal{M}^4 \subset \mathbb{R}^D$ must be a sub-manifold of the equi-potential hypersurface. This is well-known in the examples of fuzzy spaces such as S_N^2 or $\mathbb{C}P^2$ [9, 14, 15] where $X^a X_a = \text{const.} \mathbb{1}$. As a consequence, the tangential conservation law (18), which a priori is modified and reads

$$\begin{aligned} [X_a, T^{ab} + \frac{m^2}{4} [X^a, X^b]_+] &= 0 \\ [X_a, T^{ab}] &= -\frac{m^2}{4} [X_a X^a, X^b] \\ &\sim \frac{m^2}{4} i \theta^{\mu\nu} \partial_\mu x^2 \partial_\nu x^b = 0, \end{aligned} \quad (54)$$

is in fact unchanged and holds also in the presence of a potential, since $\partial_\mu V(x) = 0$ on \mathcal{M}^4 . Therefore self-dual ω with $G_{\mu\nu} = g_{\mu\nu}$ supplemented with the additional condition $x^2 = C =$

const. fulfill the modified (semi-classical) Ward identity (54) — and also the tangential conservation law Eqn. (36).

We can easily extend the analysis of the potential terms to include the next-to-leading order (n.l.o.) corrections in $\theta^{\mu\nu}$. To do this we replace the matrix product with the Groenewold-Moyal star product (cf. [2, 3]) in Darboux coordinates, where $\theta^{\mu\nu}$ is constant. Then the semi-classical limit including n.l.o. corrections of the mass term reads

$$\begin{aligned} S_m &\sim m^2 \int d^4x \sqrt{g} e^{-\sigma} \left(x^a x_a - \frac{1}{8} \theta^{\mu\nu} \theta^{\mu'\nu'} \partial_\mu \partial_{\mu'} x^a \partial_\nu \partial_{\nu'} x_a \right) \\ &\sim m^2 \int d^4x \sqrt{g} e^{-\sigma} x^a x_a, \end{aligned} \quad (55)$$

where the last line follows from partial integration, noting that $\rho = \sqrt{g} e^{-\sigma}$ is constant in Darboux coordinates. Similarly, the quartic term (51) in the semi-classical limit including n.l.o. corrections reads

$$\begin{aligned} S_{V4} &\sim \mu_4 \int d^4x \sqrt{g} e^{-\sigma} \left(2(x^2)^2 - \frac{1}{4} \theta^{\mu\nu} \theta^{\mu'\nu'} \left(x^2 \partial_\mu \partial_{\mu'} x^a \partial_\nu \partial_{\nu'} x^b \eta_{ab} + \frac{1}{2} \partial_\mu \partial_{\mu'} x^2 \partial_\nu \partial_{\nu'} x^2 \right) \right. \\ &\quad \left. - \frac{3}{8} \theta^{\mu\nu} \theta^{\mu'\nu'} \left(\frac{1}{2} \partial_\mu \partial_\rho x^2 - g_{\mu\rho} \right) \left(\frac{1}{2} \partial_\nu \partial_\sigma x^2 - g_{\nu\sigma} \right) \right) \\ &\sim \mu_4 \int d^4x \sqrt{g} \left[2e^{-\sigma} (x^2)^2 - \frac{1}{4} e^{-\sigma} \theta^{\mu\nu} \theta^{\mu'\nu'} \left(x^2 \partial_\mu \partial_{\mu'} x^a \partial_\nu \partial_{\nu'} x^b \eta_{ab} + \frac{7}{8} \partial_\mu \partial_{\mu'} x^2 \partial_\nu \partial_{\nu'} x^2 \right) \right. \\ &\quad \left. + \frac{3}{8} g^{\mu\mu'} \partial_\mu \partial_{\mu'} x^2 - \frac{3}{8} \right], \end{aligned} \quad (56)$$

$$\sim 2\mu_4 \int d^4x \sqrt{g} e^{-\sigma} (x^2)^2 \quad (57)$$

where $x^2 \equiv x^a x^b \eta_{ab}$, using again $x^2 = \text{const.}$ on solutions of the e.o.m.

However, some of the $O(X^6)$ terms considered above might modify the equation which determines $\theta^{\mu\nu}$, which should be studied elsewhere.

4 Non-Abelian sector

In this section we briefly discuss the relevance of the additional terms under consideration of the non-Abelian sector of the model, which arises on backgrounds corresponding to n coinciding branes. In order to avoid notational conflicts, we denote the basic matrices with Y^a in this section, governed by the same matrix model as above

$$S_{YM} = -\frac{\Lambda_0^4}{4g^2} \text{Tr}[Y^a, Y^b][Y^{a'}, Y^{b'}] \eta_{aa'} \eta_{bb'}, \quad (58)$$

but for a matrix background of the form

$$Y^a = \begin{pmatrix} Y^\mu \\ Y^i \end{pmatrix} = \begin{cases} X^\mu \otimes \mathbb{1}_n, & a = \mu = 1, 2, \dots, 2n, \\ \phi^i \otimes \mathbb{1}_n, & a = 2n + i, \ i = 1, \dots, D - 2n. \end{cases} \quad (59)$$

Here we also introduce a dimensionful scale parameter Λ_0 of dimension $length^{-1}$, so that $X^a \sim x^a$ acquire the appropriate dimension of length. As shown in [10], the fluctuations of the tangential resp. transversal $\mathfrak{su}(n)$ -components of Y^a lead to $\mathfrak{su}(n)$ -valued gauge fields resp.

scalar fields coupled to $G_{\mu\nu}$. The Yang-Mills matrix model (58) then describes non-Abelian gauge theory coupled to gravity, with effective gauge coupling “constant”

$$\frac{1}{g_{YM}^2} = \frac{\Lambda_0^4 e^\sigma}{g^2}, \quad (60)$$

which reduces to $g_{YM}^2 \sim g^2$ assuming $\Lambda_0 \sim \Lambda_{NC}$ as discussed above. Therefore any of the additional terms in the matrix model action discussed above also lead to additional terms for the non-Abelian gauge fields. The form of these terms is strongly restricted by gauge invariance. As usual, any higher-order terms in the field strength must be suppressed on dimensional grounds by the non-commutativity scale $\frac{1}{\Lambda_{NC}}$ (which arises from $\theta^{\mu\nu}$ or through the scalar field e^σ), and are therefore irrelevant at low energies. However, they may in general also contain explicit curvature terms. For example, the order 6 terms are expected to contain terms of type

$$S_6(F) \sim \frac{\Lambda_0^6}{\Lambda_{NC}^8} \int d^4x \sqrt{g} \operatorname{tr} \left(c_1 F^{\mu\nu} F^{\rho\sigma} R_{\mu\nu\rho\sigma} + c_2 D F D F + c_3 F[F, F] \right). \quad (61)$$

In particular, the first of these possible terms explicitly depends on the Riemann curvature and has a structure similar to the non-standard term $\theta\theta R$ which already appeared in Eqn. (40), where $F \leftrightarrow \theta$. Such terms should be expected anyway in the quantum effective action. Similarly, the $O(X^{10})$ terms (41a) leading to the Einstein-Hilbert action are expected to contain the gauge structure

$$S_{10}(F) \sim \frac{\Lambda_0^{10}}{\Lambda_{NC}^{16}} \int d^4x \sqrt{g} D F F D F F \quad (62)$$

as well as terms which are lower-order in F such as (61).

We close this section with a comment on the mass term (50). At first sight, it may appear that it would lead to a mass term for the non-Abelian gauge fields, which would be in conflict with gauge invariance. However, this is of course not the case, as is well-known in the examples of fuzzy spaces [9, 14, 15]. A careful derivation of its effect on the $\mathfrak{su}(n)$ components would require to use the 2nd order Seiberg-Witten map, which we will not carry through here.

5 Quantization and one-loop effective action

In this final section we briefly discuss the quantization of the matrix model

$$S_\Psi = -\operatorname{Tr} \left(\frac{1}{4} [X^a, X^b] [X_a, X_b] + \frac{1}{2} \bar{\Psi} \gamma_a [X^a, \Psi] \right) \quad (63)$$

and the significance of the additional terms which we have introduced above, restricting ourselves essentially to one loop.

Several different points of view can be taken. First, one can use the geometrical interpretation of the above model as an action for matter and fields on branes with non-trivial geometry $g_{\mu\nu}$, following [5, 6]. The (one-loop) induced action due to integrating out matter and fields can then be obtained from the standard heat kernel expansion on such a background, using well-known Seeley-de Wit coefficients. This leads to an induced Einstein-Hilbert action, as well as additional “non-standard” terms. Assuming an effective UV-cutoff⁶ Λ , the contribution due

⁶This could arise either by adding an explicit UV-cutoff term such as $\operatorname{Tr} \square X^a \square X_a$, or in the IKKT model [16] upon adding soft SUSY breaking terms which may lead e.g. to compactification on spontaneously generated S_N^2 [17].

to scalar fields φ in the matrix model (which arise as part of the $\mathfrak{su}(n)$ sector in the matrix model in backgrounds of the form $X^a \otimes \mathbb{1}_n$) has the expected form [8]

$$\Gamma_\varphi = \frac{1}{16\pi^2} \int d^4x \sqrt{g} \left(-2\Lambda^4 - \frac{1}{6}\Lambda^2 R[g] \right). \quad (64)$$

The induced action due to fermions is more complicated, because the matrix Dirac operator in (63) leads to a non-standard spin connection on general $\mathcal{M}^4 \subset \mathbb{R}^D$. For geometries with $g_{\mu\nu} = G_{\mu\nu}$, the result can be written as [12]

$$\begin{aligned} \Gamma_\Psi = \frac{k}{16\pi^2} \int d^4x \sqrt{g} \left[4\Lambda^4 + \Lambda^2 \left(-\frac{1}{3}R + \frac{1}{4}\partial^\mu \sigma \partial_\mu \sigma + \frac{1}{8}e^{-\sigma} \theta^{\mu\nu} \theta^{\rho\sigma} R_{\mu\nu\rho\sigma} + \frac{1}{4}\square_g x^a \square_g x_a \right) \right. \\ \left. + \mathcal{O}(\log \Lambda) \right], \end{aligned} \quad (65)$$

where k is the number of components of D -dimensional Dirac fermions. Remarkably, the terms of order Λ^2 essentially coincide with the semi-classical limit of the additional matrix model terms considered in Sections 3.1–3.2 (up to powers of e^σ which provide the required scale as in (62)). Indeed, it should be possible to perform the quantization directly within the matrix model, leading to quantum corrections to the effective action within the framework of matrix models. This should lead to a quantum effective action given by additional terms in the matrix model, such as the terms studied above. We have therefore identified the corresponding terms in such an effective matrix model, consistent with the semi-classical computation. This also provides an indirect check for the results in [12].

Furthermore, having the above matrix model terms at our disposal, we can use them as counter terms in order to cancel unwanted terms in the effective action such as $R\theta\theta$ or the “extrinsic” term $\square_g x^a \square_g x^a$. One can then indeed adjust the model such that the gravitational action reduces essentially to the Einstein-Hilbert action, plus higher-order terms which are suppressed by $\frac{1}{\Lambda}$. Therefore the matrix model can be used to realize and to quantize general relativity, or some very closely related gravity theory.

On the other hand, introducing such counter terms by hand will in general spoil the good renormalization properties of the “pure” Yang-Mills matrix model, which equivalently can be viewed as non-commutative gauge theory on \mathbb{R}_θ^4 . In particular, the IKKT model [16] (possibly with soft SUSY breaking terms such as a mass term) corresponds to $\mathcal{N} = 4$ NC SYM on \mathbb{R}_θ^4 , and is expected to be finite [18, 19]. Here the above terms should preferably be used only for intermediate steps, e.g. to introduce a controlled UV cutoff which should be removed in the end. It is quite conceivable that a realistic gravity action arises purely from the finite, induced gravitational terms below e.g. the $\mathcal{N} = 4$ breaking scale. At this point, we should briefly discuss the important aspect of UV/IR mixing NC gauge theories, and its relevance in the present context.

u(1) sector and UV/IR mixing It is well known in NC field theory that the renormalization of the $\mathfrak{u}(1)$ and the $\mathfrak{su}(n)$ sectors differ drastically at low energies⁷ due to UV/IR mixing [20, 21]: the $\mathfrak{u}(1)$ sector diverges in the IR if the UV cutoff is removed. This is usually perceived as a problem but is in fact very welcome here and consistent with the identification of the $\mathfrak{u}(1)$ sector in terms of gravity. A careful analysis from the point of view of emergent gravity [6] shows that

⁷Even though this appears to break the full gauge invariance of the matrix model, this is not the case: the $U(1)$ invariance is simply transmuted into an invariance under symplectomorphisms.

the IR divergence is precisely due to the induced gravitational action in the $\mathfrak{u}(1)$ sector. For example, the Λ^4 divergence of the cosmological constant in (64) essentially arises from the IR limit of the *effective cutoff*

$$\Lambda_{eff}^4(p) = \frac{1}{(\frac{1}{\Lambda^2} + \frac{1}{4}\frac{p^2}{\Lambda_{NC}^4})^2} \sim \Lambda^4 + \dots \quad (66)$$

in the quantum effective action. $\Lambda_{eff}^4(p)$ is in fact of order $O(1)$ at $p \approx \Lambda_{NC}$, where p is the momentum scale of the gravitational action. Hence the gravitational $\mathfrak{u}(1)$ sector scales differently under renormalization than the non-Abelian $\mathfrak{su}(n)$ sector, but is reconciled with the mild running of the non-Abelian sector above the non-commutativity scale. This suggests that taking into account the full RG flow, the strong sensitivity of the cosmological constant on the energy scale in the IR becomes mild at Λ_{NC} . A similar statement applies to the quadratically divergent Einstein-Hilbert term. It is precisely this behavior which should reconcile the apparent non-renormalizability of gravity with the good renormalization behavior of Yang-Mills gauge theory, which in the case of the IKKT model (or closely related models) is expected to be finite.

More specifically, recall that the bare cosmological constant is given by the $\mathfrak{u}(1)$ sector of the Yang-Mills term $-(2\pi)^2 \text{Tr}[X^a, X^b][X_a, X_b] \sim 4 \int d^4x \sqrt{g}$ in the matrix model⁸. It is thus quite conceivable that taking into account quantum corrections, the cosmological constant is small in the IR, but merges with the $\mathfrak{su}(n)$ Yang-Mills action at Λ_{NC} . However while it is consistent, this does not yet explain why the vacuum energy should indeed be small in the IR. At this point, it is interesting to recall that flat \mathbb{R}^4 (in fact any harmonically embedded space) is a solution also in presence of an arbitrarily large vacuum energy in the matrix model, unlike in General Relativity. It remains to be seen if this observation carries over in some way to (modified) solutions of Einstein-Hilbert type.

6 Conclusion

We have shown how gravitational actions including the Einstein-Hilbert action (with an additional scale factor) can be obtained as higher-order terms in matrix models of Yang-Mills type. The resulting actions are consistent with the induced gravitational terms in the quantum effective action of the matrix model, as obtained previously using a semi-classical heat kernel computation [12]. This exhibits the gravity sector in these matrix models more explicitly, and allows to identify and control the precise form of the gravitational action. In general, both the extrinsic and the intrinsic geometry of the space-time brane $\mathcal{M}^4 \subset \mathbb{R}^D$ play a role. However, for special cases only the intrinsic geometry enters, as in General Relativity. This allows a controlled study of the gravitational sector of the matrix model at the quantum level. It provides a new, non-perturbative and background-independent approach to quantum gravity where space and geometry emerge at low energies but are not put in by hand.

There are different avenues which can be pursued further. First, one can focus on the pure gravity sector of the (bosonic) matrix model using the additional terms introduced in this paper. At the quantum level, this would essentially require the systematic study of a suitable version of the renormalization group flow for this type of matrix model, e.g. by scaling the size of the matrices, or using an explicit cutoff term such as $\Box X^a \Box X_a$. Alternatively, one can consider finite versions of the matrix model, notably the IKKT model [16], and closely related models

⁸ assuming $g_{\mu\nu} = G_{\mu\nu}$; however additional higher-order terms in the action may also contribute.

such as [22]. In that case, the gravitational action emerges as part of the quantum effective action in a finite (non-commutative) model including gauge fields and matter. One may hope that this leads to a fully consistent quantum theory of all fundamental interactions including gravity. However, much more work is required before such a long-term goal can be achieved.

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Appendix A: Derivation of (40)

We assume $G_{\mu\nu} = g_{\mu\nu}$ throughout. The first part of this result follows immediately from the following identity (71b)

$$\square X^a \sim -\{x^b, \{x^c, x^a\}\} \eta_{bc} = -e^\sigma \square_g x^a \quad (67)$$

together with $\rho = \sqrt{g}e^{-\sigma}$. The second term is more difficult to analyze. We use the (constant) background metric η_{ab} to pull down Latin indices, i.e. $x_a \equiv x^b \eta_{ab}$, and consider first

$$\begin{aligned} & (2\pi)^2 \text{Tr}[X^c, [X^a, X^b]][X_c, [X_a, X_b]] \sim \int d^4x \sqrt{g} \partial_\rho \{x^a, x^b\} \partial^\rho \{x_a, x_b\} \\ &= \int d^4x \sqrt{g} \nabla_\rho (\theta^{\alpha\beta} \partial_\alpha x^a \partial_\beta x^b) \nabla^\rho (\theta^{\mu\nu} \partial_\mu x_a \partial_\nu x_b) \\ &= \int d^4x \sqrt{g} \left(\nabla_\rho \theta^{\alpha\beta} \nabla^\rho \theta^{\mu\nu} \partial_\alpha x^a \partial_\beta x^b \partial_\mu x_a \partial_\nu x_b + 2\theta^{\mu\nu} \nabla_\rho \theta^{\alpha\beta} \partial_\alpha x^a \partial_\beta x^b \nabla^\rho (\partial_\mu x_a \partial_\nu x_b) \right. \\ & \quad \left. + \theta^{\alpha\beta} \theta^{\mu\nu} \nabla_\rho (\partial_\alpha x^a \partial_\beta x^b) \nabla^\rho (\partial_\mu x_a \partial_\nu x_b) \right) \\ &= \int d^4x \sqrt{g} \left(\nabla_\rho \theta^{\alpha\beta} \nabla^\rho \theta^{\mu\nu} g_{\alpha\mu} g_{\beta\nu} + 4\theta^{\mu\nu} \nabla_\rho \theta^{\alpha\beta} \partial_\alpha x^a \partial_\beta x^b \partial_\mu x_a \nabla^\rho \partial_\nu x_b \right. \\ & \quad \left. + \theta^{\alpha\beta} \theta^{\mu\nu} \left(2\nabla_\rho \partial_\alpha x^a \partial_\beta x^b \nabla^\rho \partial_\mu x_a \partial_\nu x_b + 2\nabla_\rho \partial_\alpha x^a \partial_\beta x^b \partial_\mu x_a \nabla^\rho \partial_\nu x_b \right) \right). \end{aligned} \quad (68)$$

Using (30) as well as

$$g_{\mu\mu'} g_{\beta\beta'} \theta^{\mu'\beta'} = -e^\sigma \theta_{\mu\beta}^{-1}, \quad (69)$$

this simplifies as

$$\begin{aligned} & (2\pi)^2 \text{Tr}[X^c, [X^a, X^b]][X_c, [X_a, X_b]] \\ & \sim \int d^4x \sqrt{g} \left(-\nabla_\rho \theta^{\alpha\beta} \nabla^\rho (e^\sigma \theta_{\alpha\beta}^{-1}) + 2\theta^{\alpha\beta} \theta^{\mu\nu} \nabla_\rho \partial_\alpha x^a \nabla^\rho \partial_\mu x_a g_{\beta\nu} \right) \\ &= \int d^4x \sqrt{g} \left(-\nabla_\rho \theta^{\alpha\beta} \theta_{\alpha\beta}^{-1} \partial^\rho e^\sigma - e^\sigma \nabla_\rho \theta^{\alpha\beta} \nabla^\rho \theta_{\alpha\beta}^{-1} + 2e^\sigma g^{\alpha\mu} (-R_{\alpha\mu} + \square_g x^a \nabla_\alpha \partial_\mu x_a) \right) \\ &= \int d^4x \sqrt{g} \left(2e^\sigma \partial^\mu \sigma \partial_\mu \sigma - e^\sigma \nabla_\rho \theta^{\alpha\beta} \nabla^\rho \theta_{\alpha\beta}^{-1} + 2e^\sigma (-R + \square_g x^a \square_g x_a) \right) \\ &= \int d^4x \sqrt{g} e^\sigma \left(e^{-\sigma} \theta^{\mu\rho} \theta^{\eta\alpha} R_{\mu\rho\eta\alpha} - 4R + 4\partial^\mu \sigma \partial_\mu \sigma + 2\square_g x^a \square_g x_a \right), \end{aligned}$$

where the identities of Lemma 1 were used. Hence, we get

$$\begin{aligned} & (2\pi)^2 \text{Tr}[X^c, [X^a, X^b]][X_c, [X_a, X_b]] \\ & \sim 2 \int d^4x \sqrt{g} \left(\theta^{\rho\alpha} \theta^{\mu\eta} R_{\mu\rho\eta\alpha} - 2e^\sigma R + 2e^\sigma \partial^\mu \sigma \partial_\mu \sigma + e^\sigma \square_g x^a \square_g x_a \right). \end{aligned} \quad (70)$$

Lemma 1 *The following identities are useful:*

$$\nabla_\mu (e^{-\sigma} \theta^{\mu\nu}) = 0, \quad (71a)$$

$$\{x^b, \{x^c, x^a\}\} \eta_{bc} = e^\sigma \square_G x^a, \quad (71b)$$

$$\int d^4x \sqrt{g} e^\sigma g^{\rho\rho'} \nabla_\rho \theta_{\mu\alpha}^{-1} \nabla_{\rho'} \theta^{\alpha\mu} = \int d^4x \sqrt{g} e^\sigma (e^{-\sigma} \theta^{\mu\rho} \theta^{\eta\alpha} R_{\mu\rho\eta\alpha} - 2R + 2\partial^\mu \sigma \partial_\mu \sigma), \quad (71c)$$

$$R_{\lambda\mu\nu\rho} \theta^{\lambda\mu} \theta^{\nu\rho} = 2R_{\lambda\nu\mu\rho} \theta^{\lambda\mu} \theta^{\nu\rho}, \quad (71d)$$

$$\theta_{\mu\nu}^{-1} \nabla_\alpha \theta^{\mu\nu} = -2\partial_\alpha \sigma. \quad (71e)$$

Proof (71a) and (71b) was shown in [10]. To show (71c), we use the Jacobi identity and proceed as follows:

$$\begin{aligned} \int d^4x \sqrt{g} e^\sigma g^{\rho\rho'} \nabla_\rho \theta_{\mu\alpha}^{-1} \nabla_{\rho'} \theta^{\alpha\mu} &= - \int d^4x \sqrt{g} e^\sigma g^{\rho\rho'} (\nabla_\mu \theta_{\alpha\rho}^{-1} + \nabla_\alpha \theta_{\rho\mu}^{-1}) \nabla_{\rho'} \theta^{\alpha\mu} \\ &= -2 \int d^4x \sqrt{g} e^\sigma g^{\rho\rho'} \nabla_\mu \theta_{\alpha\rho}^{-1} \nabla_{\rho'} \theta^{\alpha\mu} \\ &= 2 \int d^4x \sqrt{g} e^\sigma g^{\rho\rho'} \left(\theta_{\alpha\rho}^{-1} \nabla_\mu \nabla_{\rho'} \theta^{\alpha\mu} + \theta_{\alpha\rho}^{-1} \nabla_{\rho'} \theta^{\alpha\mu} \partial_\mu \sigma \right) \\ &= 2 \int d^4x \sqrt{g} e^\sigma (g^{\rho\rho'} \theta_{\alpha\rho}^{-1} [\nabla_\mu, \nabla_{\rho'}] \theta^{\alpha\mu} + \partial^\mu \sigma \partial_\mu \sigma) \\ &= 2 \int d^4x \sqrt{g} e^\sigma \left(g^{\rho\rho'} \theta_{\alpha\rho}^{-1} (-R_{\mu\rho'\eta}{}^\alpha \theta^{\eta\mu} - R_{\mu\rho'\eta}{}^\mu \theta^{\alpha\eta}) + \partial^\mu \sigma \partial_\mu \sigma \right) \\ &= 2 \int d^4x \sqrt{g} e^\sigma (e^{-\sigma} \theta^{\alpha\rho} \theta^{\eta\mu} R_{\mu\rho\eta\alpha} - R + \partial^\mu \sigma \partial_\mu \sigma) \end{aligned} \quad (72)$$

using (71a) i.e. $\nabla_\mu \theta^{\mu\nu} = \theta^{\mu\nu} \partial_\mu \sigma$, (71d) as well as

$$\nabla^\rho \theta_{\alpha\rho}^{-1} = 0 \quad (73)$$

which holds for $G = g$. Finally (71e) follows from the fact that

$$\mathcal{J}^\mu{}_\nu := e^{-\sigma/2} \theta^{\mu\mu'} g_{\mu'\nu} \quad (74)$$

is unimodular $\det \mathcal{J} = 1$, which implies

$$\begin{aligned} 0 &= \partial_\alpha \det \mathcal{J} = (\mathcal{J}^{-1})^\mu{}_\nu \nabla_\alpha \mathcal{J}^\nu{}_\mu \\ &= e^{2\sigma} \partial_\alpha e^{-2\sigma} + g^{\mu\sigma} \theta_{\sigma\nu}^{-1} \nabla_\alpha \theta^{\nu\eta} g_{\eta\mu} \\ &= -2\partial_\alpha \sigma + \theta_{\eta\nu}^{-1} \nabla_\alpha \theta^{\nu\eta}. \end{aligned} \quad (75)$$

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